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DESCRIPTIVE GEOMETRY AND ITS MERITS AS A COLLEGIATE AS WELL AS AN ENGINEERING SUBJECT.[1]

By WILLIAM H. ROEVER.

1. The Evolution of Descriptive Geometry. To the needs of the architect on the one hand and to those of the artist on the other, can be attributed the origin and much of the development of that branch of mathematics which is now called descriptive geometry. With the development of the builder's art, there must also have developed the process of making plans and elevations of buildings. In fact, the Roman architect Vitruvius used the terms "ichnographia" and "orthographia" for plan and elevation, respectively. In the attempts to solve the problems encountered in the construction of arches and other architectural forms, the art of stereotomy (stone and wood cutting) was gradually developed in the Middle Ages. To solve these problems the objects under consideration were represented by their plans and elevations, the usefulness of which was thus extended from that of mere representation to that also of the solution of stereometric problems. A remarkable step was taken in 1738 when Frězier separated the geometric constructions of this art from their technical applications and used them to solve problems of space. But it was Gaspard Monge (1746–1818) who, by enriching and systematizing this new constructive geometry of space, elevated it to the dignity of a pure science and gave to it the name descriptive geometry. Monge also solved with it the ordinary problems of geodesy and topography.

The needs of the artist, as has already been implied, were responsible for the early development of perspective. Exhumed mural paintings from ancient

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[1] An address delivered at the third annual meeting of the Mathematical Association of America, held at the University of Chicago, December 27–28, 1917.

Rome revealed a knowledge of the vanishing point and of other notions concerning perspective. Though lost in the Middle Ages, when oblique parallel projection took its place, perspective was again discovered at the time of the Renaissance simultaneously in Italy, the Netherlands, and Germany. In the mathematical sense it was treated in the seventeenth and eighteenth centuries by men like Desargues, Taylor, Zanotti, and Lambert. In the nineteenth century its considerations brought forth the modern projective geometry at the hands of Poncelet, Möbius, Steiner, von Staudt, and others.[1]

Projective geometry made possible a deeper inquiry into the characteristics of projection than did the descriptive geometry of Monge, and hence the former science soon exerted a marked influence upon the latter. Some writers, especially in Germany, interwove projective with descriptive geometry. Besides contributing to the development of descriptive geometry through the modern projective geometry which it created, perspective may also be regarded as responsible for the development of parallel projection and axonometry. It might even be said that the oblique parallel projection of the Middle Ages was a pseudo-perspective used because of the lack of knowledge of true perspective. In the perspective coördinates of Desargues may already be gleaned notions of modern axonometry, while Lambert in his "Free Perspective" (1759) considered parallel projection as a limiting or special case. The nineteenth century witnessed many further contributions to oblique projection and axonometry at the hands of men like Möllinger, Weisbach, and Pohlke. To perspective also may be attributed the origin of photogrammetry, which has for its object the problem inverse to that of perspective, that is, the determination of the plan and elevation of an object from perspectives (*i. e.*, central projections) of that object.[2]

Of material aid in the representation of space objects by drawings in a plane are the representations of the shadows which they cast upon themselves and upon other objects, as well as the shading, *i. e.*, the reproduction of the intensity of illumination, or "brightness," of the different parts of the surface of the body represented. Thus the theories of shadows and illumination became a branch of descriptive geometry. For this and other reasons a better knowledge of the theory of surfaces was necessary. The nineteenth century witnessed in France the development of the theory of surfaces and their curvature, the theory of shadows, and, at the hands of Monge and his pupils, the theory of illumination. The latter theory was also developed in Germany and Italy.

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[1] A type of perspective for a special purpose is the stereographic projection, which was invented by Hipparchus about 140 B. C. for representing a spherical surface upon a plane, and is used to make celestial and terrestrial maps.

[2] A generalization of ordinary perspective is exhibited in what is called relief perspective, in which the half of space lying on one side of the picture plane is represented in the space between the picture plane and a plane parallel thereto. It is a remarkable fact that in 1447 when ordinary perspective was just beginning to be understood, there should have been produced such a perfect example of relief sculpture as that by Lorenzo Ghiberti in the doors of the main portal of the baptistry at Florence. This work, which Michael Angelo declared to be worthy of the Gates of Paradise, appears in general to have been constructed in accordance with the rules of relief perspective, which were not formulated until many years later.

2. The Position of Descriptive Geometry within the Realm of Mathematics. Concerning the position of descriptive geometry within the realm of mathematics, nothing that can be said will have more weight than the views of Professor Gino Loria of the University of Genoa.[1] According to this eminent geometer, mathematicians are not agreed as to the position of descriptive geometry within the entire domain of mathematical disciplines. Some (among which he does not count himself) incorporate it, following Fiedler's example, with projective geometry. Others, without denying its inner relation with the theories of modern projective geometry, regard it as amongst the applied disciplines. The latter view can, no doubt, be approved if one thinks of it only as a means to the proper construction of drawings in architecture, technology, crystallography, etc. But one is strongly inclined to abandon this view when he considers that the conceptions on which descriptive geometry rests and the methods which it continually applies are, on the whole, only such as are taught by the old Euclidean geometry and the modern projective geometry, and that throughout its entire domain the procedure is so very exact that it is comparable with analysis and algebra. In fact, descriptive geometry offers the most beautiful and instructive examples of a rigorous and complete development of several useful methods for the unambiguous representation of the whole of space upon a plane. We regard, therefore—inasmuch as we include in applied descriptive geometry the theory of illumination and shadows—the science with which we will now occupy ourselves as a province of the great realm of pure mathematics, though a region on the border thereof over which we may be transferred into the field of applied mathematics.

3. The Purposes of Descriptive Geometry. From the preceding account of the evolution of descriptive geometry it appears that the principal objects which have been sought, and in the course of time attained, are:

1. *The representation of the objects of space by means of figures which lie in a plane.*
2. *The solution of the problems of space by means of constructions which can be executed in the plane, i. e., the transformation into plane constructions* (like joining two points by a line, cutting a line by a line, and drawing a circle of given center and radius)[2] *of the graphically impossible space operations* (like passing a plane through a point and a line, cutting a line by a plane, cutting a plane by a plane, and describing a sphere of given center and radius).[3]

The criterion of what shall be meant by a good plane representative of a space object has, more or less unconsciously, been taken to be *that the plane representative, when properly placed, shall produce upon the retinal surface of the eye the same image as that produced by the object itself.*[1] That it is possible to satisfy this criterion, at least approximately, results from the following physiological facts:[2]

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[1] See preface to Vol. I, Loria, G., *Vorlesungen über Darstellende Geometrie* (1907), B. G. Teubner, Leipzig.

[2] These plane operations are sometimes called the postulates of construction.

[3] These objects may be extended so as to include: (*a*) The representation of a surface upon a plane, and the corresponding transformation of operations on the surface to those of the plane. The stereographic projection of Hipparchus, which has already been mentioned, is an example of such a representation. However, the general problem of this type properly belongs to mathematical geography; (*b*) The representation of all, or part, of space within a portion thereof, and the corresponding transformation of operations. As an example of this, relief perspective has already been mentioned.

1. All of the rays of light which emanate from a point P (Fig. 1) and pass through the pupil into the eye undergo several refractions and then unite again in a point P_n of the retinal surface of the eye. The particular ray PP_n passes through the eye practically unbroken. This ray may be taken as a substitute for the slim bundle of rays which come from P and pass through the pupil into the eye. Its direction is that in which the eye perceives the point P to lie. It is called the sight-line of P.

2. The sight-lines from the various points $P, Q, R, \cdots$ of a body Σ all pass through a fixed point K of the eye. This point is called the optical center of the eye (see Fig. 2). Hence these sight-lines belong to a bundle of lines of which the common point is K. The retinal image Σ_n, which is composed of the points $P_n, Q_n, R_n, \cdots$, is thus seen to be the geometrical intersection of the retinal surface of the eye with those lines of this bundle which come from the points of Σ.

From these two facts there results the fundamental geometrical relation which every drawing Σ' of a space object Σ must satisfy in order that it should produce upon the retinal surface of the eye the same image as that produced by the object itself. From fact 1 it follows that any light-emitting point P' which lies on the sight-line of P has the same retinal image as P. In order therefore to set up a plane picture Σ' which shall produce the same retinal image as that which is produced by Σ, it is sufficient to replace the points $P \cdots$ of Σ by the points $P' \cdots$ in which the sight-lines of the points $P \cdots$ pierce the picture plane (see Fig. 3). Since by fact 2 all of the sight-lines which come from the points of the space object Σ belong to a bundle of lines of common point K (optical center of the eye),

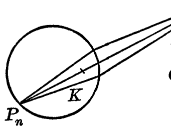

FIG. 1.

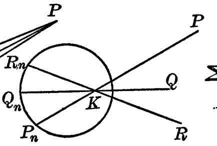

FIG. 2.

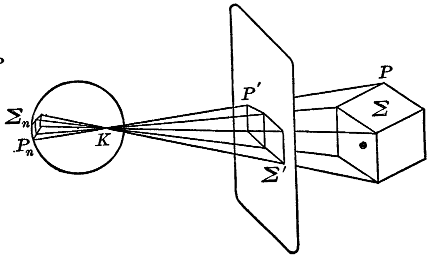

FIG. 3.

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[1] In relief perspective the criterion of good representation is that a straight line of the space to be represented shall go over into a straight line of the representing space.

[2] The remainder of this paragraph, including statements 1 and 2, has been taken, with some modifications, from A. Schoenflies, *Einführung in die Hauptgesetze der Zeichnerischen Darstellungsmethoden* (1908), B. G. Teubner, Leipzig.

it follows that the representative Σ' in the picture plane is simply the intersection of this plane with those lines of the bundle K which come from the points of Σ. In other words, *a plane representative (i. e., a picture) of a space object which will, when properly placed, produce the same retinal image as the object itself produces, must be a central projection of that object.*[1] Even when such a picture is not properly placed with respect to the eye, or when such a picture is replaced by one which is obtained by parallel instead of by converging rays (whether these rays be perpendicular or oblique to the picture plane), the eye, because of its power of accommodation, recognizes therein the object represented. Thus we are led to the conclusion that *a central or parallel (orthographic or oblique) projection of a space object is a good plane representative of that object.* These types of projection form the basis of nearly all of the methods of representation which are used in descriptive geometry.

In order to attain the second object of descriptive geometry, *i. e.*, to solve by constructions in a plane the geometric problems of space, it is essential that there shall exist an unambiguous correspondence between the elements of space (points, lines, planes, etc.) and the plane representatives of these elements. By this it is meant that it must not only be possible to pass without ambiguity from a space element to its plane representative, but that it must also be possible to pass back again without ambiguity from the plane representative to the space element. It is evident (see Figs. 4, 5, 6) that a single projection does not satisfy these requirements. For, while to each point P of space there corresponds but one point P' of the picture plane α, it is not possible to pass back again without ambiguity from P' to P, because any point whatever of the projecting ray of P could be taken as corresponding to P'. In fact, the very criterion which has been given for the good plane representation of space objects involves the ambiguity that a general point P' (Fig. 3) of the plane representative may be the picture of any point whatever of its sight-line KP'. Now, in so far as the solution of space problems is concerned, a method of representation which has the property of *unambiguous correspondence* will suffice, even if it does not satisfy the *criterion of good representation* which has been given. On the other hand we have already seen that the different types of projection satisfy the criterion of good

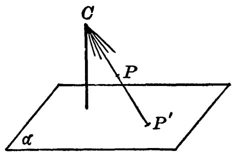

FIG. 4.

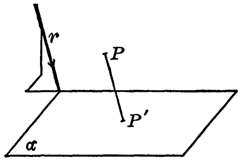

FIG. 5.

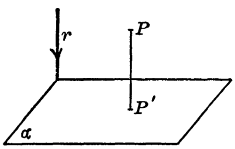

FIG. 6.

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[1] By a *central projection*, from a point C, of the points P of space upon a plane α, is meant those points P' in which the lines CP pierce the plane α (see Fig. 4). If the projecting lines instead of emanating from a point C are parallel to a straight line r, the projection is said to be *parallel* instead of central. If the line r is *not* perpendicular to α, the parallel projection is called *oblique* (see Fig. 5); while if r is perpendicular to α, the projection is called *orthographic* (see Fig. 6).

representation. But more than one projection is needed in a method of representation which shall have the property of unambiguous correspondence.[1] That which is used in addition to one projection may be another projection or it may be something else.

4. The Methods of Descriptive Geometry Exemplified. In the *Mongean Method of Double Orthographic Projection* a point P is projected orthographically upon each of two perpendicular planes. These planes, which are generally supposed to be horizontal and vertical and are therefore called respectively the *horizontal plane* π_1 and the *vertical plane* π_2, intersect in a straight line which is called the *ground line* g (see Fig. 7). The orthographic projections of P upon π_1 and π_2 are denoted by $'P$ and P'' respectively. In order to avoid the inconvenience of two planes, the plane π_1 is rotated around the ground line into coincidence with π_2, so that $'P$ then assumes in π_2 the position P'. The plane π_2, with which the revolved position of π_1 now coincides, is called the *drawing plane* (see Fig. 8). Thus we see that to the point P of space there correspond in the drawing plane the two points P' and P'' which lie on the same perpendicular to the ground line, and, conversely, to such a pair of points in the drawing plane there corresponds but one point in space. Hence this method of representation possesses the property of unambiguous correspondence. Similarly, to a line p of space there corresponds a pair of lines p' and p'' in the drawing plane, and, conversely, to any pair of lines in the drawing plane there corresponds (in general) but one line in space. Instead of being determined by its projections p' and p'', a line p may also be given by its traces P_1 and P_2, that is, by the points in which it pierces π_1 and π_2, respectively. A plane of space is represented by its traces and hence we can say that to a plane μ of space there corresponds in the drawing plane a pair of lines m_1 and m_2 which intersect on the ground line, and, conversely, to such a pair of lines of the drawing plane there corresponds but one plane of space.[2]

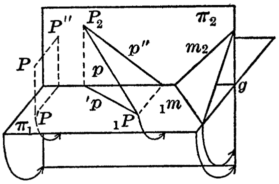

FIG. 7.

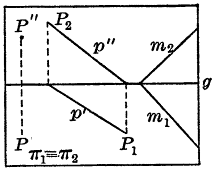

FIG. 8.

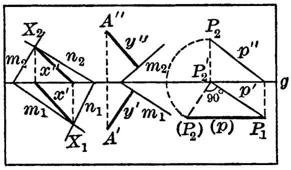

FIG. 9.

From what has been said it is evident that (1) the condition that a point be on a line is that the projections of the point lie on the corresponding projections of the line; (2) the condition that a line lie in a plane is that the traces of the line lie in the corresponding traces of the plane. It can also easily be shown that (3) the condition that a line and a plane be perpendicular to each other is that the projections of the line be perpendicular to the corresponding traces of the plane.

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[1] One is here reminded of the fact that binocular vision makes possible the perception of three-dimensional space.

[2] While $'P$ and P'' are the projections of the point P, the pair of points (P', P'') is the representative of P in the drawing plane. However, it is customary to speak of P', P'' as the projections of P. Similar remarks apply to the projection of a line and to the traces of a line and of a plane.

The last condition makes possible the solution of perpendicularity problems. To solve the metrical problems use is made of a process called *rebatting*. This consists in revolving a plane figure around the horizontal or vertical trace of its plane until it coincides with the picture plane. In Fig. 9 are given the solutions by the Mongean method of each of the three problems:

Problem I. To find the line of intersection $x \equiv (x', x'')$ of the two planes $\mu \equiv [m_1, m_2]$, $\nu \equiv [n_1, n_2]$.

Problem II. To find the line $y \equiv (y', y'')$ which passes through the point $A \equiv (A', A'')$ and is perpendicular to the plane $\mu \equiv [m_1, m_2]$.

Problem III. To find the length of that portion of the line $p \equiv (p', p'')$ which is contained between its traces P_1 and P_2.

The solution of Problem I depends upon condition 2 and the fact that line x lies in both of the given planes. The solution of Problem II depends upon conditions 1 and 3. To solve Problem III the right triangle $P_2P_2'\,{}_1P$ (Fig. 7) is revolved around p' into coincidence with π_1. Hence in Fig. 9 the segment $\overline{P_2'(P_2)}$ is drawn through P_2' perpendicular to p' and equal to $\overline{P_2'P_2}$. Thus we have solved by constructions in a plane three problems of space. Of these, I is a problem of geometry of position, II is a perpendicularity problem, and III is a metrical problem.[1]

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[1] The problems of space which involve the point, line, and plane have been divided into three groups (see Loria, *Vorlesungen*, etc., already cited): (*A*) Problems of Geometry of Position, (*B*) Perpendicularity Problems, (*C*) Metrical Problems. In each of these groups there is an indefinitely great number of problems, but they may all be solved when a few problems, called the fundamental problems of the group, can be solved.

The fundamental problems of group *A* may be taken to be:

I. To find the line connecting two points.
II. To find the line of intersection of two planes.
III. To find the plane through a point and a line.
IV. To find the point common to a plane and a line.

In I one of the points may lie at infinity, and in III either the point or the line may lie at infinity For these special positions the problems become respectively: I*a*. Through a point pass a line parallel to another line; III*a*. Through a line pass a plane parallel to another line; III*b*. Through a point pass a plane parallel to another plane. The problem: "Through a point *A* to pass a line. *x* intersecting the lines *b* and *c*" may be solved in either of the following ways: (1) Find the intersection *x* of the plane of *A* and *b* by the plane of *A* and *c*. (2) Find the point *X* where *b* pierces the plane of *A* and *c*, and then *x* is the line connecting *A* and *X*. Similarly every problem of geometry of position can be solved by a few applications of one or more of the fundamental problems.

The fundamental problems of group *B* may be taken to be:

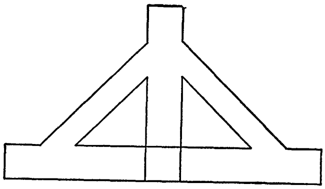
I. Through a point pass a line perpendicular to a plane.
II. Through a point pass a plane perpendicular to a line.
III. Through a point pass a line perpendicular to and touching a line.
IV. Through a line pass a plane perpendicular to a plane.
V. Find the common perpendicular to two non-intersecting lines.

The fundamental problems of group *C* may be taken to be:

I. To determine the length of the segment connecting two points.
II. To determine the magnitude of the angle formed by two lines.
III. To determine the magnitude of the angle formed by two planes.

This brief account of the Mongean method is sufficient to show its value as a means of solving graphically (with the instruments of the geometer) the problems of space. As to its value in the production of pictures, let it first be observed that most of the objects of architecture and technology contain, among their bounding surfaces, planes which are mutually perpendicular and which intersect in edges of the object. It is on these planes, or on planes parallel to these, that the objects are generally projected (orthographically) in order to obtain their Mongean representatives (*i. e.*, their pictures). Thus certain planes (namely those perpendicular to both of the planes of projection) are shown merely by lines, and therefore the Mongean pictures of such objects fail to convey a satisfactory notion of the space forms of such objects. This point is well illustrated by Figs. 10 and 11, which are the plan and elevation, respectively, and together form the Mongean representative of a bracket-shaped object. On the other hand, Fig. 12 is a projection (also orthographic) of the same object on a plane which is not parallel to one of the mutually perpendicular planes of the object. Evidently Fig. 12 conveys to the mind a much better notion of the space form of the object which it represents than do Figs. 10 and 11, from which, however, the dimensions of the object may easily be taken. It is thus evident why an architect will give the plans and elevations of a building to his contractor whereas he will show to his client a picture of the type of Fig. 12.

Fig. 10.



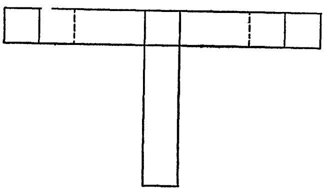


Fig. 11.

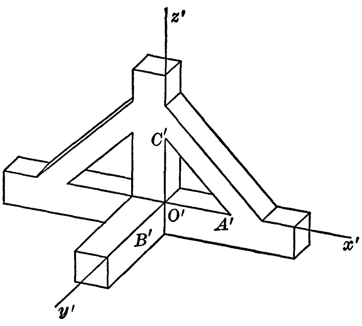


Fig. 12.

Notwithstanding the fact that Fig. 12 is not a true perspective (*i. e.*, a central projection) it produces a retinal image which does not differ much from that produced by the object itself. Such pictures, as well as true perspectives and oblique projections, can be constructed from their plans and elevations by well-known processes of the Mongean method. However, a quicker and more satisfactory method for their production is contained in the so-called *Axonometric Method*, about which something will now be said. Inasmuch as the axonometric method furnishes but a single picture, one naturally asks whether this method—which is capable of producing such good pictures—can also be used to solve graphically the problems of space. The answer to this question is that this method can so be used. In order to see this let us think of the various points of an object to be represented as referred to a system of three mutually perpendicular axes Ox, Oy, Oz (which, if the body is of the type already mentioned, may be three of its edges). Let a general point be denoted by P, its projections (orthographic) on the planes yOz, zOx, xOy by P', P'', P''', respectively, and its projections (orthographic) on the axes Ox, Oy, Oz by P_x, P_y, P_z, respectively. Then these seven points, together with the origin O, determine the so-called projecting parallelopiped $PP''P_xP'''P'P_zOP_y$ of the point P. In projecting the object by a system of parallel rays on a general plane π (plane of the paper), the axes project into the lines O^*x^*, O^*y^*, O^*z^*, respectively, and the points P, P', P'', P''', P_x, P_y, P_z project into the points P^*, P'^*, P''^*, P'''^*, $P_x{}^*$, $P_y{}^*$, $P_z{}^*$, respectively, so that the lines $P^*P'^*$, $P''^*P_z{}^*$, $P'''^*P_y{}^*$ are parallel to O^*x^*, $P^*P''^*$, $P'''^*P_x{}^*$, $P'^*P_z{}^*$ are parallel to O^*y^*, and $P^*P'''^*$, $P''^*P_x{}^*$, $P'^*P_y{}^*$ are parallel to O^*z^* (see Fig. 13). It is evident that from any one of the six pairs of points

(1) (P^*, P'^*), (P^*, P''^*), (P^*, P'''^*), (P''^*, P'''^*), (P'''^*, P'^*), (P'^*, P''^*)

the complete projection $P^*P''^*P_x{}^*P'''^*P'^*P_z{}^*O^*P_y{}^*$ of the projecting parallelopiped of the point P can be obtained. Let us suppose now that the segments $\overline{O^*A^*}$, $\overline{O^*B^*}$, $\overline{O^*C^*}$ (Fig. 13) are the projections on the picture plane of unit segments $\overline{OA}$, $\overline{OB}$, $\overline{OC}$ of the axes Ox, Oy, Oz, respectively. Then if the coördinates x, y, z, of a point P with respect to the axes Ox, Oy, Oz, are known, the projection P^* of this point upon the picture plane can easily be found by taking $\overline{O^*P_x{}^*} = x\cdot\overline{O^*A^*}$, $\overline{P_x{}^*P'''^*} = y\cdot\overline{O^*B^*}$, $\overline{P'''^*P^*} = z\cdot\overline{O^*C^*}$. However, the position of P^* alone would not be sufficient for the reversal of this process. But if any one of the six pairs of points (1) were given, the train of segments

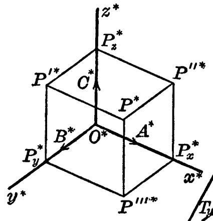


FIG. 13.

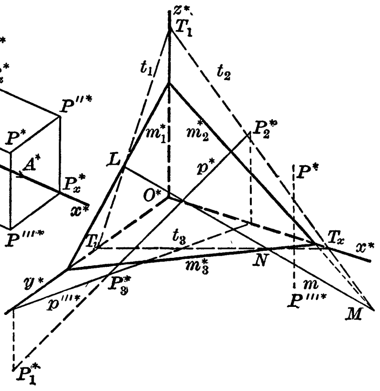


FIG. 14.

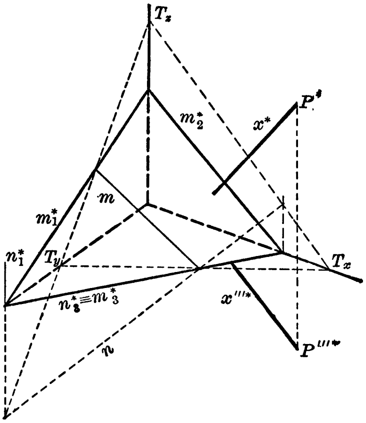


FIG. 15.

$O^*P_x^*P'''^*P^*$ would be determined and hence the coördinates of P could be read off from Fig. 13. Thus we see that in the axonometric method, as in the Mongean, it takes two points in the picture plane to represent unambiguously a point in space. Similarly, it takes two lines p^*, p'''^* (or any of five other pairs) of the picture plane to represent unambiguously a line p of space, while it takes two lines of the picture plane m_1^*, m_2^* which intersect on one of the axonometric axes O^*x^*, O^*y^*, O^*z^* to represent unambiguously a plane μ of space (see Fig. 14). Of particular interest are the traces t_1, t_2, t_3 of the picture plane which form the axonometric triangle $T_xT_yT_z$. The sides of this triangle meet the corresponding traces m_1^*, m_2^*, m_3^* of a general plane μ in the points L, M, N of a straight line m, which is called the axonometric trace of the plane μ. If, in particular, the parallel projection is orthographic, the sides of the axonometric triangle are perpendicular respectively to the axonometric axes (see Fig. 14). Statements (1), (2), (3) given under the Mongean Method are true here also if we understand by projections, *the projections on the picture plane* and by traces, *the traces in the picture plane*.[1] With this information we are in a position to solve the

Problem. To find the line $x \equiv (x^*, x'''^*)$ which shall pass through the point $P \equiv (P^*, P'''^*)$ and be perpendicular to the plane $\mu \equiv [m_1^*, m_2^*]$.

Solution for orthographic axonometry (see Fig. 15): Draw the axonometric triangle $T_xT_yT_z$ with sides perpendicular to O^*x^*, O^*y^*, O^*z^*. Then the axonometric trace of the plane μ is the line m; the perpendicular to m through P^* is x^*. Next represent a plane $\nu \equiv [n_1^*, n_3^*]$ which shall pass through m_3 and be perpendicular to the plane xOy. Then n_3^* coincides with m_3^* while n_1^* is parallel to O^*z^*. The axonometric trace of this plane is the line n. Hence since x''' lies in the plane xOy, it follows that x'''^* is perpendicular to n, and it passes through P'''^* because x must pass through P. The required line x has thus been found since its representatives in the picture x^*, x'''^* have been found.

The solution of this problem suggests, what is true, namely, that by the axonometric method, as well as by the Mongean, all the problems of space can be solved, whether these be problems of geometry of position, perpendicularity problems or metrical problems.

It was stated above that if the lengths of the three segments $\overline{O^*A^*}$, $\overline{O^*B^*}$, $\overline{O^*C^*}$, Fig. 13 (or what is just as good, numbers proportional to their lengths), were known, the coördinates of any point P which was represented in the picture plane, by any one of the six pairs of points (1), could be read off. It is evident that if the three axes Ox, Oy, Oz are equally inclined to the picture plane and the projecting rays are perpendicular to this plane, the three segments $\overline{O^*A^*}$, $\overline{O^*B^*}$, $\overline{O^*C^*}$ will be equal in length. In this case the angles $\xi = \sphericalangle y^*O^*z^*$, $\eta = \sphericalangle z^*O^*x^*$, $\zeta = \sphericalangle x^*O^*y^*$ will each be equal to 120° and then the projection is called *Isometric.* If, however, the projection still being orthographic, the three mutually perpendicular axes Ox, Oy, Oz are not equally inclined to the picture plane, there will exist between the lengths of the segments a relation which

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[1] Statement 3, thus modified, is true only for orthographic axonometry, *i. e.*, for the case where the projecting rays are perpendicular to the picture plane.

depends upon the angles ξ, η, ζ (which may be chosen subject merely to the condition that they shall be obtuse). The construction shown in Fig. 16 enables one to choose the angles ξ, η, ζ and the segments $\overline{O^*A^*}$, $\overline{O^*B^*}$, $\overline{O^*C^*}$ in their proper relations. In this construction Δ and Γ are circles of common center O^*, while $O^*(A)$ and $O^*(B)$ are perpendicular radii lying below the horizontal line which passes through O^*. Through the points where $O^*(A)$ cuts Δ and Γ, horizontal and vertical lines are drawn, respectively, meeting in A^*. In a similar manner B^* is determined. A vertical tangent t is drawn to Δ meeting Γ in (C) above O^*, and then through (C) a horizontal line is drawn cutting the vertical line through O^* in C^*. The three segments $\overline{O^*A^*}$, $\overline{O^*B^*}$, $\overline{O^*C^*}$ thus found are of proper lengths and directions.

If the projection is oblique instead of orthographic, the relation between the angles ξ, η, ζ and the segments $\overline{O^*A^*}$, $\overline{O^*B^*}$, $\overline{O^*C^*}$ depends not only upon the inclinations of the axes to the picture plane, but also upon the direction of the projecting rays. But *no matter how one may choose the angles* ξ, η, ζ *and the lengths* l, m, n *of these segments, the three directed segments* (*i. e.*, vectors) $\overline{O^*A^*}$, $\overline{O^*B^*}$, $\overline{O^*C^*}$ *of the picture plane may always be regarded as the parallel oblique projection of three mutually perpendicular concurrent space vectors of common length.* This is a statement of *Pohlke's Theorem,* which was discovered in 1853.

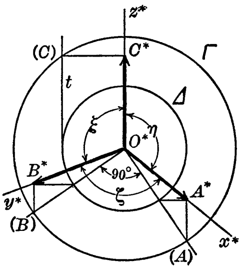

FIG. 16.

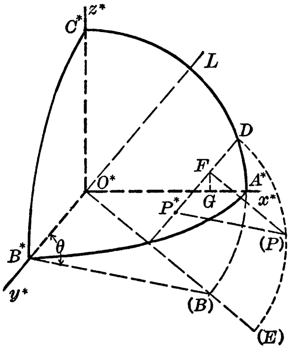

FIG. 17.

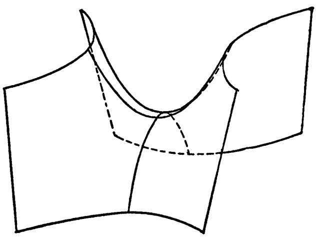

FIG. 18.

Long before the mathematical facts concerning parallel projections were known, however, it was used—often as a substitute for perspective, the rules of which were either not known or too complicated for the draughtsman. Forms of it were used in the Middle Ages before perspective was understood. The particular type of parallel projection is determined by the direction of the projecting rays and the inclination of the picture plane to the mutually perpendicular axes Ox, Oy, Oz. In the choice of the particular assumption, one is guided by practical considerations. The two particular assumptions which are commonly made are the following: *First,* the picture plane is parallel to the plane of two of the axes; *second,* the picture plane passes through the vertical axis, but is inclined to the other two. It is here assumed that the projecting rays are inclined to the picture plane; the case where they are perpendicular, having already been considered to some extent. The first type is essentially that used in the science of fortifications, and is called cavalier perspective (also military or bird perspective). The second type, and also more general types, are much used in the representation of crystals. Of the two types, the second is preferable because it coincides approximately with the middle part of perspective at a great distance, whereas in cavalier perspective there is a considerable distortion. Cavalier perspective is also used in mathematical text-books to represent space relations. Thus Figs. 4, 5, 6 and 7 of this paper are such representations. In Fig. 17 the first octant of a sphere is so represented. In the same figure is given a solution of the

Problem. The projection P^* of a point P of the sphere is given; required the coördinates of P.

Solution: A projecting ray through the point B (where the positive y-axis pierces the sphere) is projected orthographically upon the picture plane into the line O^*y^*. The plane which passes through this projecting ray BB^* and is perpendicular to the picture plane cuts from the first octant of the sphere a quadrant of a great circle. By revolving this plane around its trace y^*O^*L into the picture plane the quadrant assumes the position $LA^*(B)$ and the ray BB^* assumes the position $(B)B^*$. The angle $\theta = \sphericalangle O^*B^*(B)$ is the inclination of the projecting rays to the picture plane. Through P a plane parallel to the plane just revolved is passed. This cuts from the sphere a quadrant of a small circle, which on being turned into the picture plane (as was the large quadrant) assumes the position $D(P)(E)$. From this the revolved position (P) of P can be found by drawing $P^*(P)$ parallel to $B^*(B)$. Then the coördinate y is the perpendicular $\overline{F(P)}$ from (P) to P^*D, z is the perpendicular $\overline{FG}$ from F to O^*x^*, and x is eqaul to $\overline{O^*G}$. The meeting of DF and $O^*(B)$ on the ellipse is accidental.

In Fig. 18 a hyperbolic paraboloid is represented in cavalier perspective.

It has already been stated that parallel projection (both orthographic and oblique) was used as a substitute for perspective. It furnishes pictures which, while not producing (as does perspective) retinal images identical with those of the object, are nevertheless representations which convey to the mind adequate notions of the objects pictured. It has the advantage over perspective that its pictures may be "scaled off" and thus used as working drawings, just as the Mongean representatives. It also furnishes a means of solving graphically the problems of space.

Now, just as parallel projection was modified, or rather augmented, so as to make of it a method which can be used for the graphical solution of space problems, so central projection, or perspective, can also be thus modified. In this method, which is called that of *Free Perspective,* a line of space is represented by two points in the picture plane, namely, by its trace and its vanishing point, a plane of space is represented by its trace and its vanishing line, and a point is represented by its projection and the representative of a line or a plane on which it lies. Thus there exists an unambiguous correspondence between space and the plane, making possible the solution of space problems by this method.[1]

To solve problems concerning fortifications, military engineers use a method which is employed in representing the natural surface of the ground in topographical and hydrographical maps. It is called the *Method of Contour Lines.* In this method a point is represented by its orthographic projection on a datum plane and the number which represents its distance from this plane, a line is represented by its projection and a scale on this projection (the points of the scale being the points into which are projected the intersections of the line by equally spaced planes parallel to the datum plane), and a plane is represented by one of its lines of greatest slope. Thus is made possible the solution of space problems by this method.

It is hoped that the preceding account of the several methods of descriptive geometry will have given the reader a notion of the nature of the subject. Many things have not been mentioned, but those interested are referred to the books already mentioned.

5. Didactic Considerations. Having discussed the objects and nature of descriptive geometry, let us now consider what place it should occupy in the educational curricula. Developed from technical needs, this science was, and essentially still is, peculiar to the technical schools. It was taught and developed there almost exclusively. Germany and Austria followed France (where Monge published his lectures simultaneously with the opening of the École Polytechnique in Paris in 1795) into the field early in the nineteenth century after the founding of their technical schools. Italy did not follow until later, and, beyond the work by Brook Taylor on perspective, England took no part in the development of the subject. In Italy the most noteworthy work was done not in a technical school, but in the University of Padua by a professor of science, namely, by Bellavitis about 1851.

In this country also the teaching of descriptive geometry is confined almost exclusively to the technical schools. But the amount of time which is here devoted to the subject is so limited and the course so restricted in its scope, that the average student at the completion of his course knows hardly more than the elements of the Mongean method and very little, if anything, about other parts of the subject. Furthermore, many teachers have only a very limited knowledge of the subject. There seems to be no high requirement for a teacher of descriptive geometry as there is, for instance, for teachers of general mathematics. This being the case there is absolutely no hope of disseminating a profound and extensive knowledge of this subject.

There is a tendency in our technical schools to diminish, rather than to increase, the amount of time devoted to descriptive geometry. Therefore, if the subject is worth keeping alive—to say nothing of a hope for doing research work in it, for which there are still possibilities—it seems to me that it is the duty of our collegiate departments to offer courses in the subject and to make the scope of these courses rather extensive. Another possibility might be to raise the grade of our technical schools to that of the European technical schools, and then possibly the professors of descriptive geometry might take the interest in the subject which exists abroad. The latter alternative, *i. e.*, raising the standard of our technical schools, seems unlikely. To the former the colleges might say: "Why teach a technical subject in a college?" In reply to this, I would refer to the remarks of Professor Loria already given, namely, that descriptive geometry lies within the realm of pure mathematics but on the border thereof, so that it is an easy step to the applied. But there are other justifications for its introduction into a college curriculum. Gauss recommended it as "a nutriment-giving element to animate the real geometric spirit" [1] and Christian Wiener says, "It possesses the capacity above all to develop the power of space visualization and thus to lay the foundation for the study of higher geometry." [2]

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[1] Those interested in the methods of free perspective and contour lines are advised to read Loria's *Vorlesungen über Darstellende Geometrie,* from which much of what has been said in this paper about the Mongean and axonometric methods has been taken.

I have found that students who study analytic geometry after, or simultaneously with, descriptive geometry find the former subject more interesting and less difficult than those who have not studied the latter subject. In the study of projective geometry also, a knowledge of descriptive geometry would be of great help in making the drawings of the space relations which must necessarily be made in that subject. On the other hand, some knowledge of projective geometry is essential in the higher branches of descriptive geometry. According to Professor Loria these two subjects should be taught independently, as has been done at the University of Genoa. I consider a course in descriptive geometry in the college quite as important as one in analytic geometry and projective geometry.

Another reason for the introduction of descriptive geometry into the colleges is the almost complete ignorance concerning the subject which exists among mathematicians. This lack is generally accompanied by an inability to visualize space forms. This inability is a great handicap not only in some branches of geometry, but also in mechanics and physics, where very frequently a mental image of a situation is of as much value as refined analysis. Moreover, a general course in mathematics without graphical solutions of plane and space problems with the instruments of the geometer is almost as great an incongruity as general courses in physics and chemistry without laboratory work.

Descriptive geometry is now required for an honors degree at the University of Cambridge, England.[3] There are many advocates of its introduction into the universities of Continental Europe, and some universities actually do give courses in it. Some might hold that its introduction into a university could not be justified unless it offered problems worthy of a Doctor's thesis. I believe that it does offer such problems, and as one such, mention the theory of illumination. But even if descriptive geometry did not offer such problems, its introduction into the college, and into the university, is justifiable because of the many reasons already given.

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[1] *Encyklopädie der Mathematischen Wissenschaften*, III AB 6, page 560. This article by E. Papperitz was consulted in writing this paper.

[2] *Chr. Wiener, Lehrbuch der Darstellenden Geometrie*, Vol. I, p. 61. From this work were taken most of the historical notes of this paper.

[3] Smith, D. E. Report of Sub-Commission A of the International Commission on the Teaching of Mathematics: Intuition and experiment in mathematical teaching in secondary schools.

It is a fact that the authors of our textbooks on mathematics are deplorably ignorant concerning the making of drawings to represent space objects. Klein evidently had this in mind when he said, "Is it not as worthy an object of mathematics to be able to draw correctly as to be able correctly to calculate?" It seems only reasonable to demand that the drawings in mathematical textbooks should be made by some one who understands the underlying principles, and it would be very desirable for the author himself to be able to make his drawings.

As to the qualifications of a teacher of descriptive geometry, it would seem that he should have besides an extensive knowledge of the subject, covering the various branches above outlined, a knowledge also of plane and solid analytic geometry and projective geometry, besides the ability to make good drawings in ink as well as in pencil. The time is at hand when it is desirable to require of prospective teachers the completion of such a course. A person with such qualifications could be attached to either a department of mathematics or a department of drawing. If to the former, he would certainly be required to know more mathematics than the minimum just laid down, and if to the latter, he would probably be required to be proficient in other branches of drawing.

While this address was being prepared for publication, death claimed that one person who by many was regarded as the best teacher of descriptive geometry which this country ever had. I had the good fortune to be his pupil, and it was he who inspired me with that love for the subject which it is my earnest desire to transmit to others. I refer to the late Dr. E. A. Engler, at one time professor of mathematics at Washington University.

GEOMETRY FOR JUNIORS AND SENIORS.[1]

By E. B. STOUFFER, University of Kansas.

An examination of the list of papers presented to the American Mathematical Society during the last five years shows that on the average not more than one paper in six is concerned essentially with geometry. If the papers presented to the Society are assumed to picture approximately the research activity of America, it may well be asked whether geometry is receiving the attention which it deserves. Is a ratio of one to five or six a healthy sign for mathematics as a whole in this country?

It is true, there is no fundamental distinction between geometry and analysis—

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[1] Read at the second summer meeting of the Association, Cleveland, Ohio, September 7, 1917.